Solving Incomplete Information General Equilibrium Models in Sequence Space

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Models with information frictions are often used in macroeconomics to explain the sluggish adjustment of economic variables observed in empirical data. Despite their relevance, these models are difficult to solve due to the higher-order beliefs - agents must form expectations about others' expectations. Conventional approaches, such as Kalman filtering, require repeated iterations, leading to computational inefficiency.

This note proposes a streamlined approach to solving such models in sequence space, reducing the computation to a single matrix inversion and enabling faster, more practical implementation. The main idea is to observe that any equilibrium variable can be represented as an $MA(\infty)$ process of the exogenous shocks. Thus, together with analytical formula for the expectation of the exogenous shocks, a simple linear system can be constructed to solve for the impulse response of the equilibrium variables to the exogenous shocks.

In this note, I first present a simple model with incomplete information in Section 1 and derive the analytical solution in Section 2 using the technique in Angeletos and Huo (2021). Then, I show how to represent the model in sequence space in Section 3 and provide a numerical example in Section 4 to illustrate the computational simplicity of the proposed method.

1 Model

Consider the framework in Angeletos and Huo (2021), suppose the economy is populated by a continuum of agents indexed by $i \in [0,1]$. Each agent is affected by an demand shock, e.g. interest rate shock, denoted as ξ_t.[1] The demand shock is unobserved directly. Instead, each agent observes a signal of the shock s_{it} and forms expectations about ξ_t and y_t. The information structure is given by

$$\xi_t = \rho \xi_{t-1} + \epsilon_t \quad \text{and} \quad \epsilon_t \sim N(0,1) \tag{1}$$
$$s_{it} = \xi_t + \eta_{it} \quad \text{and} \quad \eta_{it} \sim N(0,\sigma^2) \tag{2}$$

The optimal consumption of agent i is given by a permanent income consumption function[2]

$$c_{it} = -\beta \left[\mathbf{E}_{it}[\xi_t] + \sum_{k=1}^{\infty} \beta^k \mathbf{E}_{it}[\xi_{t+k}] \right] + (1-\beta) \left[\mathbf{E}_{it}[y_t] + \sum_{k=1}^{\infty} \beta^k \mathbf{E}_{it}[y_{t+k}] \right] \tag{3}$$

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[1]This is the setup in Angeletos and Huo (2021). This is different from some HANK model that the shock and income enter to the individual problem in current term instead of expectation. My method applies to general setups

[2]For now, this assumes away the wealth effect for simplicity. The method can be extended easily to include a budget constraint.

The aggregate output is assumed to be demand-driven

$$y_t = c_t = \int_0^1 c_{it} di \tag{4}$$

This implies that the equilibrium output is given by

$$y_t = -\beta \left[\bar{\mathbf{E}}_t[\xi_t] + \sum_{k=1}^{\infty} \beta^k \bar{\mathbf{E}}_t[\xi_{t+k}] \right] + (1-\beta) \left[\bar{\mathbf{E}}_t[y_t] + \sum_{k=1}^{\infty} \beta^k \bar{\mathbf{E}}_t[y_{t+k}] \right] \tag{5}$$

where $\bar{\mathbf{E}}_t[.]$ denotes the average expectation, $\bar{\mathbf{E}}_t[.] = \int_0^i \mathbf{E}_{it}[.]di$.

The Equation 1, Equation 2, and Equation 5 define the rational expectation equilibrium of the model. One can iterate on Equation 5 to obtain the equilibrium output, which involves composition of average expectation operator, $\bar{\mathbf{E}}_t[\bar{\mathbf{E}}_{t+1}...[.]]$, or also known as the higher-order beliefs. In the absence of complete information, the law of iterated expectations does not hold for the average expectation operator. Intuitively, one solution is to evaluate Equation 5 directly by repeatedly applying the average expectation operator. However, this means that many Kalman filtering iterations are required, which is computationally expensive.

2 Analytical Solution

Using the technique in Angeletos and Huo (2021), the analytical solution to the model is given by the following

$$y_t = \frac{\Delta b^*}{(1-\theta\mathbf{L})(1-\rho\mathbf{L})} \epsilon_t \tag{6}$$

where $\mathbf{L}$ is a lag operator and θ and Δ are solution of some polynomial equation with both $\theta < 1$ and $\Delta < 1$.[3] $\frac{b^*}{(1-\rho L)}\epsilon_t$ is the solution for complete information equilibrium.

While this analytical solution is a powerful tool to draw insights from the model, it is not always available for more complicated problems. For example, one may consider a model that allows heterogeneity in β and saving. The analytical solution for this problem is not available. Thus, a lot of these models are solved numerically.

3 Sequence space Representation

The idea of sequence space representation is to find a linear system where the solution of the system is the impulse response function, or equivalently the coefficients of an $MA(\infty)$ process. Let $\mathbf{h}(x)$ be an infinite dimensional vector, where the t-th element is the impulse response of variable x_t to a shock to ϵ_0. Hence,

$$\mathbf{h}(x) = \begin{bmatrix} \frac{dx_0}{d\epsilon_0} & \frac{dx_1}{d\epsilon_0} & \frac{dx_2}{d\epsilon_0} & \cdots \end{bmatrix}'$$

Our goal is to solve for $\mathbf{h}(y)$, the impulse response of y_t to a shock to ϵ_0. Taking derivative of Equation 5 against the shock ϵ_0, we obtain

$$\frac{dy_t}{d\epsilon_0} = -\beta \sum_{k=0}^{\infty} \beta^k \frac{d\bar{\mathbf{E}}_t[\xi_{t+k}]}{d\epsilon_0} + (1-\beta) \sum_{k=0}^{\infty} \beta^k \frac{d\bar{\mathbf{E}}_t[y_{t+k}]}{d\epsilon_0}$$

[3] See Angeletos and Huo (2021)

Collecting the equations across time, we have

$$\mathbf{h}(y) = -\beta \begin{bmatrix} m'\mathbf{W}_0(\xi) \\ m'\mathbf{W}_1(\xi) \\ m'\mathbf{W}_2(\xi) \\ \vdots \end{bmatrix} + (1-\beta) \begin{bmatrix} m'\mathbf{W}_0(y) \\ m'\mathbf{W}_1(y) \\ m'\mathbf{W}_2(y) \\ \vdots \end{bmatrix} \tag{7}$$

where m is a vector of discounting factor $m = \begin{bmatrix} 1 & \beta & \beta^2 & \cdots \end{bmatrix}'$ and $\mathbf{W}_t(x)$ is the derivative of time-t expectation of variable x in the future, given by $\mathbf{W}_t(x) = \begin{bmatrix} \frac{d\bar{\mathbf{E}}_t[x_0]}{d\epsilon_0} & \frac{d\bar{\mathbf{E}}_t[x_1]}{d\epsilon_0} & \frac{d\bar{\mathbf{E}}_t[x_2]}{d\epsilon_0} & \cdots \end{bmatrix}'$. Our goal is to connect $\mathbf{W}_t(\xi)$ and $\mathbf{W}_t(y)$ to the impulse response of ξ_t and y_t.

In the following, I show the connection between $\mathbf{W}_t(x)$ and $\mathbf{h}(x)$. Let $h_{y,k}$ be the k-th element of $\mathbf{h}(y)$, the time-k response of y_t to a shock to ϵ_0. Recall that coefficients of an $MA(\infty)$ process is exactly the impulse response function. Thus, for each individual agent i, we have

$$\mathbf{E}_{i,t}[y_{t+k}] = h_{y,0} \underbrace{\mathbf{E}_{i,t}[\epsilon_{t+k}]}_{=0} + h_{y,1} \underbrace{\mathbf{E}_{i,t}[\epsilon_{t+k-1}]}_{=0} + \cdots + h_{y,k}\mathbf{E}_{i,t}[\epsilon_t] + h_{y,k+1}\mathbf{E}_{i,t}[\epsilon_{t-1}] + \cdots$$

$E_{i,t}[\epsilon_{t+k}]$ is zero for $k > 0$ because the future shock is uncorrelated with the information set at time t. Taking average across agents and differentiating against ϵ_0, we have

$$\frac{d\bar{\mathbf{E}}_t[y_{t+k}]}{d\epsilon_0} = h_{y,k}\frac{d\bar{\mathbf{E}}_t[\epsilon_t]}{d\epsilon_0} + h_{y,k+1}\frac{d\bar{\mathbf{E}}_t[\epsilon_{t-1}]}{d\epsilon_0} + \cdots$$

This implies that the vector of expectations of future y_t could be written as

$$\mathbf{W}_t(y) = \begin{bmatrix} \frac{d\bar{\mathbf{E}}_t[y_t]}{d\epsilon_0} \\ \frac{d\bar{\mathbf{E}}_t[y_{t+1}]}{d\epsilon_0} \\ \frac{d\bar{\mathbf{E}}_t[y_{t+2}]}{d\epsilon_0} \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{d\bar{\mathbf{E}}_t[\epsilon_t]}{d\epsilon_0} & \frac{d\bar{\mathbf{E}}_t[\epsilon_{t-1}]}{d\epsilon_0} & \frac{d\bar{\mathbf{E}}_t[\epsilon_{t-2}]}{d\epsilon_0} & \cdots \\ 0 & \frac{d\bar{\mathbf{E}}_t[\epsilon_t]}{d\epsilon_0} & \frac{d\bar{\mathbf{E}}_t[\epsilon_{t-1}]}{d\epsilon_0} & \cdots \\ 0 & 0 & \frac{d\bar{\mathbf{E}}_t[\epsilon_t]}{d\epsilon_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\equiv \mathbf{M}_t^\epsilon} \mathbf{h}(y)$$

The $\mathbf{M}_t^\epsilon$ matrix is an upper triangular Toeplitz matrix, formed by the derivative of the expectation of the past and current shock against the initial shock. Intuitively, since y_t follows an $MA(\infty)$ process, the initial shock ϵ_0 affects the expectation of y_t through the expectation of the current and past shock. The same logic applies to ξ_t as well. The system of equations can be written as

$$\mathbf{h}(y) = -\beta \begin{bmatrix} m'\mathbf{M}_0^\epsilon \\ m'\mathbf{M}_1^\epsilon \\ m'\mathbf{M}_2^\epsilon \\ \vdots \end{bmatrix} \mathbf{h}(\xi) + (1-\beta) \begin{bmatrix} m'\mathbf{M}_0^\epsilon \\ m'\mathbf{M}_1^\epsilon \\ m'\mathbf{M}_2^\epsilon \\ \vdots \end{bmatrix} \mathbf{h}(y)$$

The terms in the matrix $\mathbf{M}_t^\epsilon$ can be calculated analytically. Since the signal s_{it} and shock process ξ_t are exogenous, Wiener-Hopf filter can be applied to obtain the exact values of the terms in $\mathbf{M}_t^\epsilon$. In particular, the individual expectation of the current and past shock is given by

$$\mathbf{E}_{i,t}[\epsilon_{t-k}] = \frac{\lambda\sigma^2}{\rho}\left(\mathbf{L}^k + \lambda\mathbf{L}^{k-1} + \cdots + \lambda^k\right)\frac{s_{it}}{(1-\rho\mathbf{L})(1-\lambda\mathbf{L})} \tag{8}$$

where $\mathbf{L}$ is the lag operator, $\mathbf{L}x_t = x_{t-1}$, and λ is a constant given by $\lambda = \frac{1}{2}\left(\rho + \frac{1}{\rho}(1+\frac{1}{\sigma^2}) - \sqrt{(\rho + \frac{1}{\rho}(1+\frac{1}{\sigma^2}))^2 - 4}\right)$. Intiutively, the signals in the far past have less impact on the current expectation. This explains why there is a geometric decay in the weights for the past signals. However, since the expression is evaluating the expectation of a past shock at time $t-k$, the optimal filter should apply more weights to k-time past signals. This explains the presence of the λ^k term in the front. Taking average across agents, we have

$$\bar{\mathbf{E}}_t[\epsilon_{t-k}] = \frac{\lambda\sigma^2}{\rho}\left(\mathbf{L}^k + \lambda\mathbf{L}^{k-1} + \cdots + \lambda^k\right)\left(1 + \lambda\mathbf{L} + \lambda^2\mathbf{L}^2 + ...\right)\epsilon_t \tag{9}$$

The coefficient on the ϵ_{t-s} is the response of $\bar{\mathbf{E}}_t[\epsilon_{t-k}]$ to a shock to ϵ_{t-s}, i.e. $\frac{d\bar{\mathbf{E}}_t[\epsilon_{t-k}]}{d\epsilon_{t-s}}$. Recall that our goal is to find the impulse response of $\mathbf{E}_t[\epsilon_{t-k}]$ to ϵ_0. Set $s = t$, we have $\frac{d\bar{\mathbf{E}}_t[\epsilon_{t-k}]}{d\epsilon_0}$. Hence, we can fill up the matrix $\mathbf{M}_t^\epsilon$ by evaluating the coefficients of Equation 9 for $k = 0, 1, 2, ...$. This seems to be taxing, but the calculation involves only polynomial multiplication, which can be done efficiently in many programming libraries.

Finally, to solve for the impulse response of y_t to a shock to ϵ_0, we can simply invert the linear system. Since ξ_t follows an $AR(1)$ process, the impulse response of ξ_t to a shock to ϵ_0 is given by

$$\mathbf{h}(\xi) = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots \end{bmatrix}'$$

The impulse response of y_t to a shock to ϵ_0 is finally given by

$$\mathbf{h}(y) = -\left(\mathbf{I} - (1-\beta)\begin{bmatrix} m'\mathbf{M}_0^\epsilon \\ m'\mathbf{M}_1^\epsilon \\ m'\mathbf{M}_2^\epsilon \\ \vdots \end{bmatrix}\right)^{-1} \beta \begin{bmatrix} m'\mathbf{M}_0^\epsilon \\ m'\mathbf{M}_1^\epsilon \\ m'\mathbf{M}_2^\epsilon \\ \vdots \end{bmatrix} \mathbf{h}(\xi)$$

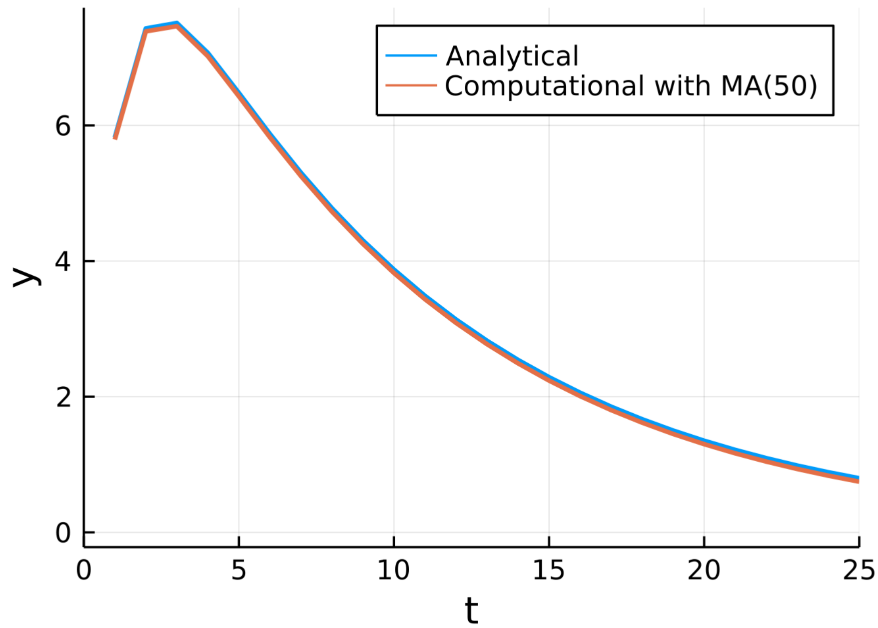
where $\mathbf{I}$ is the identity matrix. Hence, to solve for the impulse response of y_t to a shock to ϵ_0, we only need to invert a single matrix. This is computationally easier than applying repeated Kalman filtering. This method could be extended to more complex problems without significant modification. For example, one may consider a model with saving and multiple permanent types. The method can be extended to include an intertemporal budget constraint and multiple types of agents, where the aggregate policy function may take a different form. Even then, the method of connecting the impulse response of the expectation of the equilibrium variable to the impulse response of the variable itself remains the same.

4 Numerical Example

In this section, I provide a numerical example to illustrate the computation of this method. Since the impulse response function is of infinite length, a truncation is necessary to obtain numerical results. In this section, I compare the impulse response function obtained from the proposed method to the analytical solution in Angeletos and Huo (2021).

In this example, I consider an expansionary shock to ξ (i.e. $\epsilon_0 = -1$). With a truncation of 50 periods, the proposed method provides a good approximation to the analytical solution.

Figure 1. Impulse Response Function from Analytical Solution and Proposed Method



Note: The figure shows the impulse response function of y_t to a shock to $\epsilon_0 = -1$. The blue line is the impulse response function obtained from the analytical solution in Angeletos and Huo (2021). The red line is the impulse response function obtained from the proposed method with truncation set to 50.

References

ANGELETOS, G.-M. and HUO, Z. (2021). Myopia and Anchoring. *American Economic Review*, **111** (4), 1166–1200.